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Point perturbations in constant curvature spaces

Sergio Albeverio · Vladimir A. Geyler ·
Evgeniy N. Grishanov · Dmitriy A. Ivanov

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Abstract Point perturbations of the free Hamiltonian in two- and three-dimensional spaces of constant curvatures are considered. The study of the spectral properties of perturbed Hamiltonian and various asymptotics for its point levels are presented. It is shown that the binding energy in comparison with the case of zero curvature reduces in the case of Lobachevsky plane and rises in the case of 2D-sphere, when the scattering length is much less than the curvature radius.

Keywords Laplace–Beltrami operator · Point perturbations · Constant curvature spaces

1 Introduction

Last decade curved nanostructures began to be studied intensively in nanoelectronics. The ability to produce arbitrarily shaped or curved two-dimensional electron gases in semiconductors using recent developments in technology, for example regrowth of III–V semiconductors on patterned or ached substrates, opens the possibility of investigating not only the behavior of electrons in a curved quasi-two-dimensional space and the effects of varying that curvature, but also presents a novel way of investigating electron transport properties in a non-uniform transverse high magnetic field. This problem is considered in [1].

Energy spectrum and ballistic transport of 2D electrons placed on the cylindrical surface are theoretically considered in [2]. The model for the description of the spectra of electrons,

V.A. Geyler is deceased.

S. Albeverio
Bonn University, Regina-Pacis-Weg 3, Bonn 53113, Germany
e-mail: albeverio@uni-bonn.de

E.N. Grishanov · D.A. Ivanov (✉)
Mordovian State University, Bol'shevistskaya 68, Saransk 430005, Russia
e-mail: ivdmal@mail.ru

E.N. Grishanov
e-mail: grishanoven@math.mrsu.ru

holes, and excitons confined in semiconductor quantum dots based on quantum mechanics in spaces of constant curvature is suggested in [3]. In this work the comparison of the model with the experiment is carried out for CdSe and InAs quantum dots. Charged particle living on Lobachevsky plane and interacting with homogeneous magnetic field perpendicular to the plane and a point interaction are considered in [4].

The model of a particle confined to a thin curved layer in a Euclidean space are used in a lot of works, devoted to related problems (see, e.g. [5–8]). In [8] sufficient conditions which guarantee the existence of geometrically induced bound states in curved quantum layers of constant width was found. But for some cases simpler model can be used. If the thickness of the considered structure is not comparable with its size we can assume that the electron can moved only on the surface of the structure (like two-dimensional electron gas on the plate [9] and on the sphere [10]). The mathematical model of this type is used in our paper. Also we consider structures with constant curvature which are not embedded in another curved space.

One of the most interesting problems is the investigation of the spectrum of the Hamiltonian for a curved nanostructure with impurities. Particularly, it is very important to look for the dependence of the spectrum on the surface curvature and the strength of the perturbation. Physical aspects of the problem are discussed in [11]. Moreover, perturbations of three-dimensional structures can be considered as a perturbation of the system geometry.

In this paper we construct the asymptotics in the curvature for levels of the point perturbations (point levels) in the spaces of constant curvatures and investigate the properties of the point spectrum of the perturbations using this asymptotics. The zero-range potential model is basic for our considerations. It is well-developed for spaces with zero curvature (see, e.g. [12]). It should be mentioned that the first correct mathematical description of the model in the framework of the theory of a self-adjoint extensions of symmetric operators was given in [14]. We use Krein's technique to construct self-adjoint extension which gives us the model in question. Krein formula leads to the following expression for the Green function of the extension ($G_\alpha(x,y;z)$) from a one-parameter family of extensions of the Laplace operator:

$$G_\alpha(x,y;z) = G^0(x,y;z) - [Q(z,a)-\alpha]^{-1} G^0(x,q;z) G^0(q,y;z). \tag{1}$$

Here α is the extension parameter, $G^0(x,y;z)$ is the Green function of the unperturbed operator, z is the spectral parameter, $Q(z,a)$ is the Krein Q-function (regular part of the Green function) depending on the radius a of space curvature. Particularly, the energy levels (point levels) are determined as solutions of the following equation:

$$Q(z,a) - \alpha = 0, \tag{2}$$

In our case α is related with the scattering length λ:

$$\alpha = \frac{\ln\lambda}{2\pi}, \quad \text{if } d = 2, \tag{3}$$

$$\alpha = -\frac{1}{4\pi\lambda}, \quad \text{if } d = 3. \tag{4}$$

Here d is the space dimension.

The plan of the paper is as follows. We consider four spaces: Lobachevsky plane, Lobachevsky space, two-dimensional and three-dimensional spheres. For each space, at first, we investigate how the Q-function depends on a, λ and z. Here we use the fact that an explicit expression for the unperturbed Green function for these spaces is known [15]. The second step is to reveal the dependence of the energy levels on a and λ. The graphs of these dependencies for each spaces are obtained. The peculiarities of the cases of curved spaces in comparison with the Euclidian case are described. Namely, it is shown that the binding energy in comparison with the case of zero curvature reduces in the case of the Lobachevsky plane and rises in the case of the 2D-sphere, when the scattering length is much less than the curvature radius.

2 Preliminaries

Below we consider a simply connected complete Riemannian manifold X of constant curvature. Our concern is only with dimensions $d=2$ (surfaces) and $d=3$ (spaces of the Friedmann-type cosmological models). Note, that in the cases of higher dimensions operator $-\Delta_{LB}$ with the domain $C_0^\infty(X\setminus q)$ is essential self-adjoint [16]. Denoting by dl^2 the squared element of length,

$$dl^2 = g_{\alpha\beta}dx^\alpha dx^\beta, \tag{5}$$

we have for the Riemann-Christoffel tensor of curvature:

$$R_{\alpha\beta\gamma\delta} = k(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}), \tag{6}$$

where the constant quantity k is the sectional curvature of X (for $d=2$ it coincides with the Gauss curvature K). The scalar curvature R is related to k by the following formula

$$R = d(d-1)k, \quad d=2,3. \tag{7}$$

In the case of $R\neq 0$ it is more convenient to use the quantity a instead of R, where a (the curvature radius), $a\equiv a(R)>0$, is defined by

$$a = \frac{1}{\sqrt{|k|}}. \tag{8}$$

Let Δ_{LB} be the Laplace–Beltrami operator,

$$\Delta_{LB} \equiv \mathrm{divgrad} = g^{-1/2}\frac{\partial}{\partial x^\alpha}g^{1/2}g^{\alpha\beta}\frac{\partial}{\partial x^\beta}, \tag{9}$$

where as usually $g=\det\|g_{\alpha\beta}\|$. Then the Hamiltonian $H^0\equiv H^0(R)$ of a free particle of mass M on the manifold X has the form (see e.g., [17]):

$$H^0 = -\frac{\hbar^2}{2M}\left(\Delta_{LB} - \frac{d-1}{4d}R\right). \tag{10}$$

Here we suppose that the domain of H^0 is $L^2(X,\mu)$, where μ is Riemannian–Lebesgue measure on X.

As a rule, we will use the system of units such that $\hbar=1$, $M=1/2$, therefore, the energy has dimension $[L^{-2}]$, where L is a length.

Formally speaking, the perturbation H of H^0 by a point (zero-range) potential supported at a point q, $q \in X$, has the form

$$H = H^0 + \varepsilon\delta_q(x), \tag{11}$$

where the coupling constant ε should be considered as infinitely small [12, 14, 18, 19].

More precisely, adding a point perturbation potential $\varepsilon\delta_q(x)$ is equivalent to a boundary condition at the point q [12, 20, 21]. This condition is defined as follows. Let S_q be the restriction of H^0 to the domain

$$\mathcal{D}_q = \{f \in \mathcal{D}(H^0) : f(q) = 0\} \tag{12}$$

($\mathcal{D}_q$ is well defined since in the case $d \le 3$ each function from $\mathcal{D}(H^0)$ is continuous). It is easy to show that S_q is a closed symmetric operator with deficiency indices $(1, 1)$ and the deficiency subspace $\mathcal{N}_z = \ker(S_q^* - z)$ is generated by the function $x \mapsto G^0(x, q; z)$, where $G^0(x, y; z)$ is the Green function of H^0, i.e. the integral kernel of the resolvent $(H^0 - z)^{-1}$ [22]. It is well known that $G^0(x, y; z)$ is defined on the set $\{(x, y) \in X \times X : x \ne y\}) \times (C \setminus \operatorname{spec}(H^0))$; it is infinitely smooth with respect to (x, y) and analytic with respect to z. Moreover, $G^0(x, y; z)$ has the following representation

$$G^0(x, y; z) = F_0(x, y) + F_1(x, y; z), \tag{13}$$

where the function $F_1(x, y; z)$ is continuous in $X \times X$ with respect to (x, y) and F_0 has the form

$$F_0(x, y) = -\frac{1}{2\pi}\ln\rho(x, y), \quad \text{if } d = 2;$$

$$F_0(x, y) = \frac{1}{4\pi}\rho(x, y)^{-1}, \quad \text{if } d = 3.$$

Here and below $\rho(x, y)$ is the geodesic distance between points x and y.

Using (13) we see that each function f from $\mathcal{N}_z$ has the following representation

$$f(x) = a(f)F_0(x, q) + b(f) + r(x), \tag{14}$$

where $a(f),\ b(f) \in \mathbb{C}$ and $r(x) \to 0$ as $x \to q$. Then for each $\alpha \in \mathbb{R}$ the boundary condition

$$b(f) = \alpha a(f) \tag{15}$$

defines a self-adjoint extension $H_\alpha \equiv H_\alpha(R, q)$ of S_q which is distinct from $H^0(R)$ ($H^0(R)$ can be obtained from (15) by putting formally $\alpha = \infty$). The operator H_α is called the point perturbation of H^0 with a zero-potential of strength α.

Using the scattering length λ (see Introduction) we will write $H^\lambda(R, q)$ instead of $H_\alpha(R, q)$; if $\lambda = 0$ we return to the unperturbed operator H^0. At fixed R and λ all the operators $H^\lambda(R, q)$ are unitarily equivalent because the group of isometries of X is transitive. In particular, the spectrum of $H^\lambda(R, q)$ is independent of q, whereas the corresponding eigenfunctions depend on q, of course.

The Green function $G_\alpha(x, y; z)$ of H_α (also denoted by $G^\lambda(x, y; z)$ if we use the notation H^λ) can be found explicitly by means of Krein resolvent formula. Denote by $Q(z)$ so-called the Krein $\mathcal{Q}$-function: $Q(z) = F_1(q, q; z)$ (see (13)). It is clear that for the considered manifolds X, $Q(z)$ is independent of q, but depends, generally speaking, on R (i.e., on a), and we will write $Q(z;R)$ instead of $Q(z)$ if we want to take into account this dependence. The expression for the Green function of H_α is given in Introduction (1).

It is well known that $Q'(E)>0$ if $E\in\mathbb{R}\setminus\operatorname{spec}(H^0)$. Therefore, on each interval (a,b) (bounded or not) lying in $\mathbb{R}\setminus\operatorname{spec}(H^0)$ the equation

$$Q(E)-\alpha=0 \tag{16}$$

has at most one solution $\mathcal{E}$ which is, automatically, simple eigenvalue of H^λ, called a point level of H^λ. It is easy to show that the normalized eigenfunction $\Psi_\mathcal{E}$ of the level $\mathcal{E}$ has the form:

$$\Psi_\mathcal{E}(x)=c_0G_0(x,q;\mathcal{E}), \tag{17}$$

where c_0 is the normalizing constant,

$$c_0=(Q'(\mathcal{E}))^{-1/2}. \tag{18}$$

It is known (see, e.g., [12]) that

$$Q(z;0)=\begin{cases}-\frac{1}{2\pi}[\ln\sqrt{-z}-\ln 2+\gamma], \quad \text{if } d=2; -\frac{\sqrt{-z}}{4\pi}, \text{ if } d=3.\end{cases} \tag{19}$$

Here γ is the Euler constant ($\gamma=-\Gamma'(1)$) and we took the continuous branch of the square root $\sqrt{z}$ such that $\sqrt{z}>0, \forall z\in\mathbb{R}, z>0$.

If $R=0$, then the following propositions concerning point levels take place

(GS1) *Let* $d=2$. *Then for each* $\alpha\in\mathbb{R}$ (*i.e. for each* $\lambda\in(0,+\infty)$) *equation* (16) *has a unique negative solution* $\mathcal{E}^\lambda\equiv\mathcal{E}^\lambda(0)$, *which is the ground state of* $H^\lambda(0)$. *The explicit form of* $\mathcal{E}^\lambda$ *is as follows*:

$$\mathcal{E}^\lambda=-4e^{-2\gamma}\lambda^{-2}=-4e^{-2\gamma}\exp(-4\pi\alpha). \tag{20}$$

(GS2) *Let* $d=3$. *Then equation* (*16*) *has a solution if and only if* $\alpha<0$ (*i.e.* $\lambda>0$). *This solution* (*denoted by* $\mathcal{E}^\lambda(0)\equiv\mathcal{E}^\lambda$ *also*) *is unique and has the form*

$$\mathcal{E}^\lambda=-16\pi^2\alpha^2=-\lambda^{-2}. \tag{21}$$

Therefore, the ground state of $H^\lambda(0)$ *is equal to* $\mathcal{E}^\lambda$, *if* $\lambda>0$ *and to* 0 *if* $\lambda\le 0$.

We now turn to the systematic consideration of the cases $d=2$, $d=3$ and $R>0$, $R<0$.

3 The Case of Noncompact X: $R<0$

If $R<0$, then X is the d-dimensional Lobachevsky space, which is local isomorphic, in its turn, to the d-dimensional pseudo-sphere with the radius $a=\sqrt{d(1-d)/R}$. It is convenient to treat the Lobachevsky space as the Poincaré upper half-space $\mathbb{H}^d_a$ in $\mathbb{R}^d$:

$$\mathbb{H}^d_a=\{x=(x_1,\dots,x_d)\in\mathbb{R}^d : x_d>0\} \tag{22}$$

with the metric

$$dl^2 = \left(\frac{a}{x_d}\right)^2 \sum_{j=1}^{d} dx_j^2. \tag{23}$$

3.1 Lobachevsky Plane: $R < 0$, $d = 2$

In this case $R = -2/a^2$, and the Laplace–Beltrami operator Δ_{LB} has the form

$$\Delta_{LB} = \left(\frac{x_2}{a}\right)^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right). \tag{24}$$

Therefore,

$$H^0(R) = -\left(\frac{x_2}{a}\right)^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) - \frac{1}{4a^2}. \tag{25}$$

The spectrum of H^0 is purely absolutely continuous and coincides with the half-line $[0, \infty)$ (see, e.g., [13]). The Green function $G^0(x, y; z)$ has the form [15]:

$$G^0(x, y; z) = \frac{\Gamma^2(s)}{4\pi\Gamma(2s)} \sigma^{-s} F(s, s; 2s; \sigma^{-1}), \tag{26}$$

where $F(a, b, c; z)$ is the hypergeometric function, $\Gamma(s)$ is the Euler Γ-function and

$$\begin{aligned} s &\equiv s(z) = \frac{1}{2} + a\sqrt{-z}, \\ \sigma &\equiv \sigma(x, y) = \cosh^2 \frac{\rho(x, y)}{2a} = \frac{1}{2}\left(1 + \cosh \frac{\rho(x, y)}{a}\right). \end{aligned} \tag{27}$$

To find $Q(z; R)$ we use the following series expansion [23, 2.10(12)]:

$$F(a, a; 2a; z) = \frac{\Gamma(2a)}{\Gamma^2(a)} \sum_{n=0}^{\infty} \frac{((a)_n)^2}{(n!)^2} [h_n - \ln(1 - z)](1 - z)^n, \tag{28}$$

where $-\pi < \arg(1 - z) < \pi$, $(a)_n = a(a + 1)\dots(a + n - 1) = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol, $h_n = 2\psi(n + 1) - 2\psi(a)$. Here $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function. Using (28), we get

$$G^0(x, y; z) = -\frac{1}{2\pi}\left[\ln \frac{\rho(x, y)}{2a} + \psi(s) + \gamma\right] + O(\rho^2(x, y) \ln \rho(x, y)) \tag{29}$$

as $\rho(x, y) \to 0$. Therefore,

$$Q(z; R) = -\frac{1}{2\pi}\left[\psi\left(\frac{1}{2} + a\sqrt{-z}\right) - \ln(2a) + \gamma\right]. \tag{30}$$

Here we need to pay attention to the dimension of quantity Q. As in the Euclidean case, Q contains the item $\ln L$, where L is length. It is more correct to have a dimensionless value as an argument of the logarithm. Taking into account (3) we see, that the left side of the dispersion equation (16) comes to the following more correct form:

$$\tilde{Q}(z;R) = Q(z;R) - \alpha = -\frac{1}{2\pi}\left[\psi\left(\frac{1}{2}+\frac{a}{\lambda}\sqrt{-\lambda^2 z}\right) - \ln(2a/\lambda) + \gamma\right].$$

instead of $Q(z;R)$ for plotting figures.

3.1.1 Properties of the Q-Function

Let us determine the behavior of $Q(z;R)$ as $z \to 0$ and $\mathrm{Re}z \to -\infty$. Since $\psi(\frac{1}{2}) = -\gamma - \ln 4$, we have

$$\lim_{z\to 0} Q(z;R) = \frac{1}{2\pi}\ln(8a). \tag{31}$$

On the other hand

$$\psi(z) = \ln z - \frac{1}{2}z^{-1} - \frac{1}{12}z^{-2} + \frac{1}{120}z^{-4} + O(z^{-6}), \tag{32}$$

where $|\arg z| < \pi$ [23, 1.19(7)]. Therefore,

$$\begin{aligned} Q(z;R) &= -\frac{1}{2\pi}\left[\ln(-z)^{1/2} + \gamma - \ln 2 + \frac{1}{24}(a\sqrt{-z})^{-2} - \frac{7}{960}(a\sqrt{-z})^{-4}\right] \\ &\quad + O((a\sqrt{-z})^{-6}) \\ &= \frac{1}{2\pi}\ln(a) - \frac{1}{2\pi}\left[\gamma - \ln 2 + \ln(a\sqrt{-z}) + \frac{1}{24}(a\sqrt{-z})^{-2} - \frac{7}{960}(a\sqrt{-z})^{-4}\right] \\ &\quad + O((a\sqrt{-z})^{-6}) \end{aligned} \tag{33}$$

as $a\sqrt{-z} \to \infty$, $z \notin \mathrm{spec}(H^0)$. In particular, if $z \notin \mathrm{spec}(H^0)$, then

$$\lim_{R\to 0} Q(z;R) = -\frac{1}{2\pi}\left[\ln\sqrt{-z} + \gamma - \ln 2\right] = Q(z;0). \tag{34}$$

The behavior of the Q-function is described by the next lemma.

Theorem 1

(1) *$Q(z;R)$ is an increasing function of $z \in (-\infty, 0)$ at fixed $R < 0$.*
(2) *$Q(z;R)$ is a decreasing function of $R < 0$ at fixed $z \in (-\infty, 0)$.*
(3) *At the points $z = -l^2/a^2$, where l is an integer or a half-integer, $Q(z;R)$ have the next simple form*

$$Q\left(-\frac{(2n+1)^2}{4a^2};R\right) = -\frac{1}{2\pi}\left(1+\frac{1}{2}+\cdots+\frac{1}{n} - \ln(2a)\right), \tag{35}$$

$$Q\left(-\frac{n^2}{a^2};R\right) = -\frac{1}{\pi}\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1} - \frac{1}{2}\ln(8a)\right), \tag{36}$$

where $n \in N$.

Proof (1) Consider the behavior of the function $\partial Q/\partial z$. First we note that

$$\frac{\partial}{\partial z}Q(z;R)=\frac{a}{4\pi\sqrt{-z}}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}+a\sqrt{-z}\right)^{-2}. \tag{37}$$

Equation (37) shows that $\partial Q/\partial z$ is an increasing function of z on $(-\infty,0)$; hence, Q is a convex function of $z\in(-\infty,0)$. Moreover, if $z\in(-\infty,0)$, then

$$\lim_{z\to-\infty}\frac{\partial}{\partial z}Q(z;R)=0, \tag{38}$$

$$\lim_{z\to 0}\frac{\partial}{\partial z}Q(z;R)=+\infty. \tag{39}$$

(2) As to the behavior of $Q(z;R)$ as a function of $R=-2/a^2$, then we have

$$\frac{\partial}{\partial a}Q(z;R)=\frac{\sqrt{-z}}{2\pi}\left[\frac{1}{a\sqrt{-z}}-\sum_{n=0}^{\infty}\left(n+\frac{1}{2}+a\sqrt{-z}\right)^{-2}\right]. \tag{40}$$

Let us show that $\partial Q/\partial a>0$ on $(-\infty,0)$. It is sufficient to show that for each $y>0$ we have

$$\frac{1}{y}>\sum_{n=0}^{\infty}\left(n+\frac{1}{2}+y\right)^{-2}.$$

For this purpose we note that

$$\frac{1}{y}=\sum_{n=0}^{\infty}\left[\frac{1}{n+y}-\frac{1}{n+1+y}\right]$$

$$=\sum_{n=0}^{\infty}\frac{1}{(n+y)^2+n+y}>\sum_{n=0}^{\infty}\left(n+\frac{1}{2}+y\right)^{-2}.$$

Therefore, at fixed $z\in(-\infty,0)$, the function $Q(z;R)$ is an increasing function of a and hence a decreasing function of R.

(3) The necessary expressions follows from the next relation [23, 1.7(10)]:

$$\psi(z+n)=\frac{1}{z}+\frac{1}{z+1}+\cdots+\frac{1}{z+n-1}+\psi(z), \tag{41}$$

where $n\in N$. □

$\tilde{Q}(E;R)$ as a function of λ^2E and a/λ, respectively, is plotted on Figs. 1a and 1b. Recall, that λ^2E and a/λ are dimensionless.

3.1.2 Behavior of the Point Levels

To analyze the behavior of the function $\mathcal{E}^{\lambda}(R)$ it is convenient sometimes to rewrite equation (16)

$$Q(E;a)-\alpha=0$$

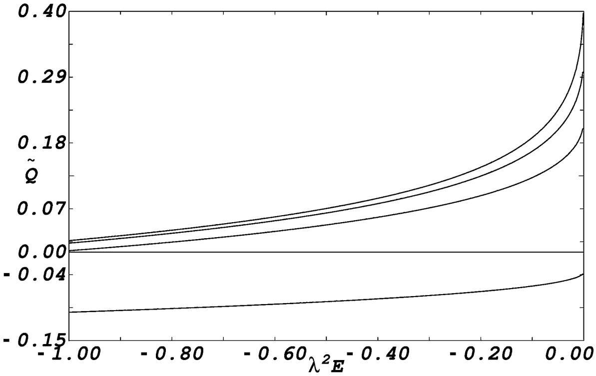


Fig. 1a $\tilde{Q}$ as a function of $\lambda^2 E$. Curves are plotted for $a/\lambda = 2, 1, 0.5, 0.1$ (in *top-down* direction)

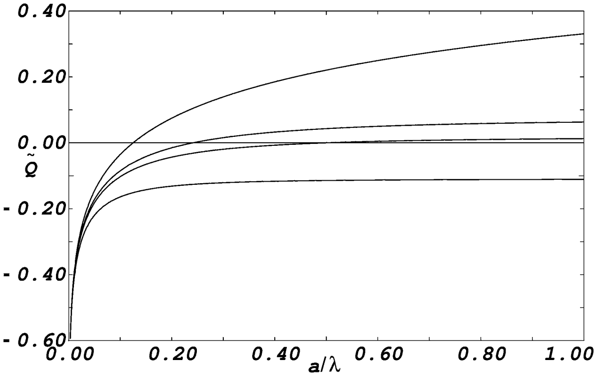


Fig. 1b $\tilde{Q}$ as a function of a/λ. Curves are plotted for $\lambda^2 E = 0, -0.5, -1, -5$ (in *top-down* direction)

in the form

$$\psi\left(\frac{1}{2} + a\sqrt{-E}\right) + \gamma = \ln\frac{2a}{\lambda}, \tag{42}$$

where the both sides are dimensionless.

The next theorem is followed from the results of the previous section.

Theorem 2

(1) *The point level* $\mathcal{E}^\lambda(R)$ *of the perturbed operator* H_α (H^λ) *exist if and only if* $\alpha < \frac{1}{2\pi}\ln(8a)$ $(\lambda < 8a)$.
(2) $\mathcal{E}^\lambda(R)$ *increases from* $-\infty$ *to* 0 *and is concave on the interval* $(0, 8a)$ *at fixed* $R < 0$.
(3) $\mathcal{E}^\lambda(R)$ *decreases from* 0 *to* $\mathcal{E}^\lambda(0)$ *as* a *runs through* $(\lambda/8, +\infty)$ (*or, which is the same, increases from* $\mathcal{E}^\lambda(0)$ *to* 0 *as* R *increases from* 0 *to* $128/\lambda^2$) *at fixed* $\lambda > 0$.

Proof (1) Equations (31) and (38) show that at fixed R, $Q(z; R)$ strictly increases from $-\infty$ to $\frac{1}{2\pi}\ln(8a)$, if z increases on the real line from $-\infty$ to 0. Therefore, (16) has a solution if and only if $\alpha < \frac{1}{2\pi}\ln(8a)$, or in other words if and only if $\lambda < 8a$.

(2) Equation (16) immediately shows that at fixed R, $\mathcal{E}^\lambda(R)$ increases from $-\infty$ to 0 as α runs through $(-\infty, \frac{1}{2\pi}\ln(8a))$ or, which is the same, as λ runs through $(0, 8a)$. Moreover,

Fig. 2a $\lambda^2\mathcal{E}$ as a function of a/λ (λ is fixed)

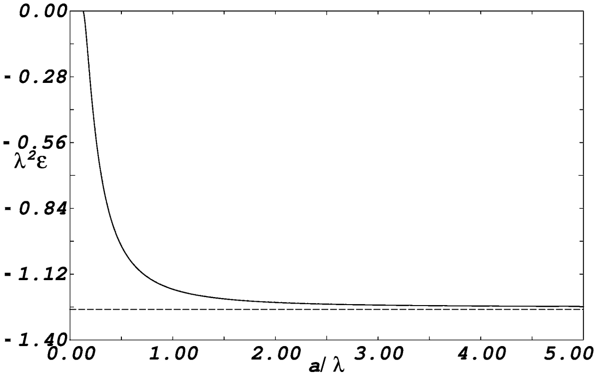
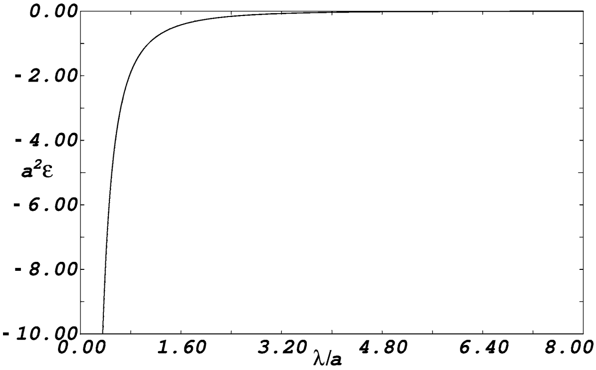


Fig. 2b $a^2\mathcal{E}$ as a function of λ/a (a is fixed)



from

$$\frac{\partial \mathcal{E}^{\lambda}(R)}{\partial \lambda} = \frac{1}{2\pi\lambda}\left(\frac{\partial Q}{\partial z}\right)^{-1} > 0 \tag{43}$$

and

$$\frac{\partial^2 \mathcal{E}^{\lambda}(R)}{\partial \lambda^2} = -\left[\frac{1}{2\pi\lambda^2} + \frac{\partial^2 Q}{\partial z^2}\left(\frac{\partial \mathcal{E}^{\lambda}(R)}{\partial \lambda}\right)^2\right]\left(\frac{\partial Q}{\partial z}\right)^{-1} < 0 \tag{44}$$

we see that at fixed R the function $\lambda \mapsto \mathcal{E}^{\lambda}(R)$ is increasing and concave on the interval $(0, 8a)$.

(3) This statement follow from the results of the previous section by the obvious way. □

The plots of functions $\lambda^2\mathcal{E}^{\lambda}(a/\lambda)$ and $a^2\mathcal{E}^{\lambda}(\lambda/a)$ are shown on Figs. 2a and 2b respectively. As noted in the previous section these quantities are also dimensionless.

The asymptotic behavior of the point perturbed levels $\mathcal{E}^{\lambda}(R)$ is described by the next theorem.

Theorem 3

(1) *If* $\lambda \ll a$ *then* $a^2\mathcal{E}^{\lambda}(R) \to -\infty$. *Moreover,*

$$\mathcal{E}^{\lambda}(R) = \mathcal{E}^{\lambda}(0) + \frac{1}{12}a^{-2} - \frac{e^{2\gamma}}{360}\lambda^2 a^{-4} + O(\lambda^4 a^{-5}). \tag{45}$$

(2) *If* $\lambda \sim 8a$ *then* $\mathcal{E}^\lambda(R) \sim 0$ *and*

$$\mathcal{E}^\lambda(R) = -\frac{4}{\pi^4 a^2}\ln^2\frac{8a}{\lambda} + O\left(\ln^3\frac{8a}{\lambda}\right), \tag{46}$$

as $8a/\lambda \to 1+0$

(3) *If* $\lambda \sim 2a$ *then*

$$\mathcal{E}^\lambda(R) = -\frac{1}{4a^2} + \frac{6}{\pi^2 a^2}\ln\frac{\lambda}{2a} + O\left(\ln^2\frac{\lambda}{2a}\right), \tag{47}$$

as R *is fixed and* $\lambda \to 2a$ *and*

$$\mathcal{E}^\lambda(-2/a^2) = -\frac{1}{\lambda^2} - \frac{2}{\pi^2\lambda^3}(12-\pi^2)(2a-\lambda) + O\big((2a-\lambda)^2\big), \tag{48}$$

as λ *is fixed and* $a \to \lambda/2$.

Proof (1) First part of this case is immediately clear if a is fixed and $\lambda \to 0$. Fix now λ and let $a \to \infty$. If there exists a sequence a_n, $a_n \to \infty$ such that the sequence $a_n\sqrt{\mathcal{E}_0^\lambda(2/a_n^2)}$ is bounded, then (30) shows that $Q(\mathcal{E}^\lambda(R_n); R_n) \to \infty$; this contradicts to the equation $Q(\mathcal{E}^\lambda(R_n); R_n) = \alpha$.

To get (45) we use asymptotics (33). We denote for simplicity $E = \mathcal{E}^\lambda(R)$, $E_0 = \mathcal{E}^\lambda(0)$ and note that

$$E = E_0\exp(4\pi q(\mathcal{E}^\lambda(R))), \tag{49}$$

where $q(z) = Q(z; R) - Q(z; 0)$. Indeed, substituting $z = E\exp(-4\pi q(\mathcal{E}^\lambda(R)))$ in $Q(z; 0)$, we get $Q(E; R)$ which is equal to α, hence, $E\exp(-4\pi q(\mathcal{E}^\lambda(R)))$ is equal to E_0. Using (33), we obtain

$$q(E) = \frac{1}{2\pi}\left(\frac{1}{24a^2 E} + \frac{7}{960a^4 E^2}\right) + O(|a^2 E|^{-3}). \tag{50}$$

Therefore,

$$E = E_0(1 + 4\pi q + 8\pi^2 q^2 + O(q^3)), \tag{51}$$

in particular, $a^2 E_0 \to -\infty$ and $E/E_0 \to 1$ as $\lambda \to 0$ or $a \to \infty$. Further,

$$E = E_0\left(1 + \frac{1}{12a^2 E} + \frac{13}{720a^4 E^2}\right) + O(|a^2 E|^{-3}), \tag{52}$$

hence,

$$E = E_0 + \frac{1}{12a^2} + \frac{1}{90a^4 E_0} + O(a^{-5}|E_0|^{-2}). \tag{53}$$

Equation (45) follows immediately from (53).

(2) Clearly that $\mathcal{E}^\lambda(R) \sim 0$ and the necessary expression follows from the asymptotics

$$Q(z; R) = -\frac{\pi a}{4}\sqrt{-z} + \frac{1}{2\pi}\ln(8a) + O(a^2 z), \tag{54}$$

as $a^2 z \to 0$ which is simply follows from the relations

$$\psi\left(\frac{1}{2}\right) = -\gamma - \ln 4,$$

$$\psi'\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{-2} = \frac{\pi^2}{2}.$$

(3) If $\lambda = 2a$, then clearly $\mathcal{E}^{\lambda}(R) = -(4a^2)^{-1}$. Using the Taylor expansion and the relation

$$\psi'(1) = \sum_{n=0}^{\infty}(n+1)^{-2} = \frac{\pi^2}{6},$$

we get the required expressions. □

In particular, (45) shows that the binding energy $|\mathcal{E}^{\lambda}(R)|$ has the estimate $|\mathcal{E}^{\lambda}(R)| \simeq |\mathcal{E}^{\lambda}(0)| - (12a^2)^{-1} + O(a^{-4})$ which is less then binding energy $|\mathcal{E}^{\lambda}(0)|$ in the case of zero curvature.

In addition, we can consider at fixed R the following collection of α for which the solution of (16) is known explicitly:

$$\alpha_n = \frac{1}{2\pi}\left(\ln(2a) - 1 - \frac{1}{2} - \cdots - \frac{1}{n}\right),$$

or

$$\lambda_n = 2a \exp\left(-1 - \frac{1}{2} - \cdots - \frac{1}{n}\right).$$

In this case $\mathcal{E}^{\lambda_n}(R) = -\frac{(2n+1)^2}{4a^2}$. On the other hand, if

$$\alpha_n = \frac{1}{\pi}\left(\frac{1}{2}\ln(8a) - 1 - \frac{1}{3} - \cdots - \frac{1}{2n-1}\right),$$

or

$$\lambda_n = 8a \exp\left(-1 - \frac{1}{3} - \cdots - \frac{1}{2n-1}\right),$$

then $\mathcal{E}^{\lambda_n}(R) = -n^2/a^2$.

3.2 Lobachevsky Space: $R < 0$, $d = 3$

In this case $R = -6/a^2$ and

$$H^0 = -\Delta_{LB} - \frac{1}{a^2}, \tag{55}$$

where Δ_{LB} now has the following form in the standard coordinates of the half-space $\mathbb{H}_a^3$:

$$\Delta_{LB} = \left(\frac{x_3}{a}\right)^2\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{1}{x_3}\frac{\partial}{\partial x_3}\right).$$

Fig. 3a λQ as a function of $\lambda^2 E$ (λ is fixed)

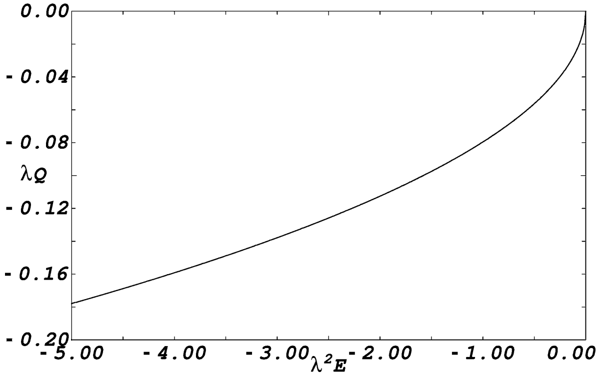
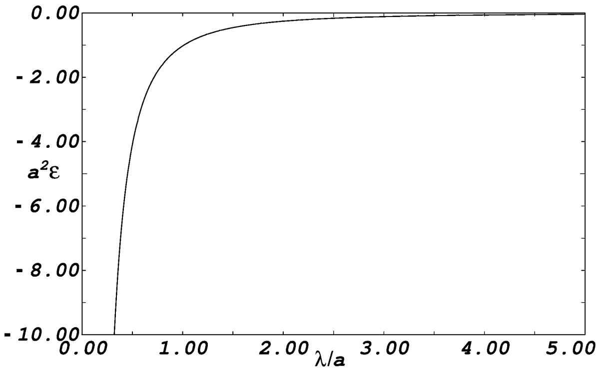


Fig. 3b $\lambda^2\mathcal{E}$ as a function of λ/a (a is fixed)



The spectrum of H^0 is purely absolutely continuous and coincides with the half-line $[0, \infty)$ (see, e.g., [13]). The Green function has the form

$$G^0(x, y; z) = \frac{\exp(-\rho(x, y)\sqrt{-z})}{4\pi a \sinh \frac{\rho(x,y)}{a}}. \tag{56}$$

Further we formulate the next simply theorem which complete our considerations at this case.

Theorem 4

(1) *For each* $R < 0$ $Q(z; R) = Q(z; 0) = -\frac{\sqrt{-z}}{4\pi}$.
(2) *Point levels* $\mathcal{E}(R)$ *exist if and only if* $\alpha < 0$ *and* $\mathcal{E}(R) = \mathcal{E}(0) = -16\pi^2\alpha^2$.

So, this case is absolute coincide with the Euclidean case.

The plots of the dimensionless functions $\lambda Q(\lambda^2 E)$ and $\lambda^2\mathcal{E}(\lambda/a)$ are shown on Figs. 3a and 3b, respectively.

4 The Case of Compact X: $R > 0$

If $R > 0$, then X is isomorphic to the d-dimensional sphere $\mathbb{S}_a^d$ of radius $a = \sqrt{d(d-1)/R}$.

4.1 Two-dimensional Sphere: $d = 2$, $R > 0$

In this case $R = 2/a^2$, where $a > 0$, and X can be considered as the standard 2-sphere $\mathbb{S}^2_a$ of radius a lying in space $\mathbb{R}^3$. In the standard polar coordinates

$$\begin{cases} x_1 = a\sin\theta\cos\varphi, \\ x_2 = a\sin\theta\sin\varphi, \\ x_3 = a\cos\theta, \end{cases}$$

where $0 < \theta < \pi$, $0 < \varphi < 2\pi$, we have

$$\begin{aligned} dl^2 &= a^2(\sin^2\theta d\varphi^2 + d\theta^2), \\ \Delta_{LB} &= \frac{1}{a^2}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right). \end{aligned} \tag{57}$$

It is well known (see, e.g., [24]) that $-\Delta_{LB}$ (the angular momentum operator) has the purely discrete spectrum consisting of levels $\frac{1}{a^2}l(l+1)$ which are degenerate with multiplicity $2l+1$. Therefore the spectrum of H^0,

$$H^0 = -\Delta_{LB} + \frac{1}{4a^2},$$

consists of the levels

$$E^0_l = \frac{1}{a^2}\left(l + \frac{1}{2}\right)^2, \quad l = 0, 1 \dots,$$

which are degenerate with the same multiplicity $2l+1$. The Green function $G_{LB}(x, y; z)$ of $-\Delta_{LB}$ is well known [15]:

$$G_{LB}(x, y; z) = \frac{1}{4\cos(\frac{\pi}{2}\sqrt{\frac{1}{4} + a^2 z})}\mathcal{P}_{-\frac{1}{2}+\sqrt{\frac{1}{4}+a^2 z}}\left(-\cos\frac{\rho(x, y)}{a}\right),$$

where $\mathcal{P}_\nu(x)$ is the Legendre function. Therefore

$$\begin{aligned} G^0(x, y; z) &= \frac{1}{4\cos(\pi a\sqrt{z})}\mathcal{P}_{-\frac{1}{2}+a\sqrt{z}}\left(-\cos\frac{\rho(x, y)}{a}\right) \\ &= \frac{1}{4\cos(\pi a\sqrt{z})}F\left(\frac{1}{2} - a\sqrt{z}, \frac{1}{2} + a\sqrt{z}; 1; \cos^2\frac{\rho(x, y)}{a}\right), \end{aligned} \tag{58}$$

where $F(a, b; c; z)$ is the Gauss hypergeometric function. Using the asymptotics

$$\mathcal{P}_\nu(x) = \frac{\sin\pi\nu}{\pi}\left[\ln\frac{1+x}{2} + 2\psi(1+\nu) + 2\gamma + \pi\cot\pi\nu\right] + O\left((x+1)\ln(x+1)\right)$$

as $x \to -1$ (see [23, 3.9(15) and 2.10(12)]), we get immediately

$$\begin{aligned} &G^0(x, y; z) \\ &\quad = -\frac{1}{2\pi}\left[\ln\rho(x, y) + \psi\left(\frac{1}{2} + a\sqrt{z}\right) + \gamma - \frac{\pi}{2}\tan(\pi a\sqrt{z}) - \ln 2a\right] \\ &\qquad + O(\rho^2(x, y)\ln\rho(x, y)) \end{aligned} \tag{59}$$

as $\rho(x,y)\to 0$. From (59) we obtain the following expression for the Krein $\mathcal{Q}$-function

$$Q(z;R)=-\frac{1}{2\pi}\left[\psi\left(\frac{1}{2}+a\sqrt{z}\right)-\frac{\pi}{2}\tan(\pi a\sqrt{z})-\ln 2a+\gamma\right], \tag{60}$$

or, using [23, 1.7(12)],

$$Q(z;R)=-\frac{1}{4\pi}\left[\psi\left(\frac{1}{2}+a\sqrt{z}\right)+\psi\left(\frac{1}{2}-a\sqrt{z}\right)-2\ln 2a+2\gamma\right]. \tag{61}$$

In this case also as in the case of Lobachevsky plane, the expression (61) contains the item of the form $\ln L$, where L is a length (see the note after (30)). Thus, in this section we use the following more correct value $\tilde{Q}$ instead of Q for plotting figures:

$$\begin{aligned}\tilde{Q}(z;R)&=Q(z;R)-\frac{\ln\lambda}{2\pi}\\&=-\frac{1}{4\pi}\left[\psi\left(\frac{1}{2}+\frac{a}{\lambda}\sqrt{\lambda^2 z}\right)+\psi\left(\frac{1}{2}-\frac{a}{\lambda}\sqrt{\lambda^2 z}\right)-2\ln(2a/\lambda)+2\gamma\right].\end{aligned}$$

Before analyzing of the spectrum of H^λ, we note that if $R\to 0$, then $Q(z;R)$ tends to the Krein $\mathcal{Q}$-function of the free Hamiltonian on the Euclidean plane. More precisely, if $E<0$, then

$$Q(E;R)\to -\frac{1}{2\pi}[\ln\sqrt{-E}-\ln 2+\gamma] \tag{62}$$

as $R\to 0$. It is sufficient to show that at $E<0$ and $x,y\in\mathbb{R}^2$, $x\neq y$, we have

$$G^0(x,y;E)\to\frac{1}{2\pi}K_0(\sqrt{-E}\rho(x,y)),$$

where $K_0(x)$ is the MacDonald function (note, that $\frac{1}{2\pi}K_0(\sqrt{-E}|x-y|)$ is just the Green function of $-\Delta$ on the plane $\mathbb{R}^2$). Indeed, using the formula

$$\mathcal{P}_\nu(-x)=\mathcal{P}_\nu(x)\cos\pi\nu-\frac{2}{\pi}\mathcal{Q}_\nu(x)\sin\pi\nu$$

(see [23, 3.4(14)]), we get

$$\begin{aligned}&G^0(x,y;z)\\&\quad=\frac{1}{4}\tan(\pi a\sqrt{z})\mathcal{P}_{-\frac{1}{2}+a\sqrt{z}}\left(\cos\frac{\rho(x,y)}{a}\right)+\frac{1}{2\pi}\mathcal{Q}_{-\frac{1}{2}+a\sqrt{z}}\left(\cos\frac{\rho(x,y)}{a}\right).\end{aligned} \tag{63}$$

Let $E<0$, then $\tan(\pi a\sqrt{E})=-i\tanh(\pi a\sqrt{-E})$, hence,

$$\begin{aligned}&G^0(x,y;E)\\&\quad=\frac{-i}{4}\tanh(\pi a\sqrt{-E})\mathcal{P}_\nu\left(\cos\left(\frac{\nu}{a}\frac{\rho(x,y)}{\nu}\right)\right)+\frac{1}{2\pi}\mathcal{Q}_\nu\left(\cos\left(\frac{\nu}{a}\frac{\rho(x,y)}{\nu}\right)\right),\end{aligned} \tag{64}$$

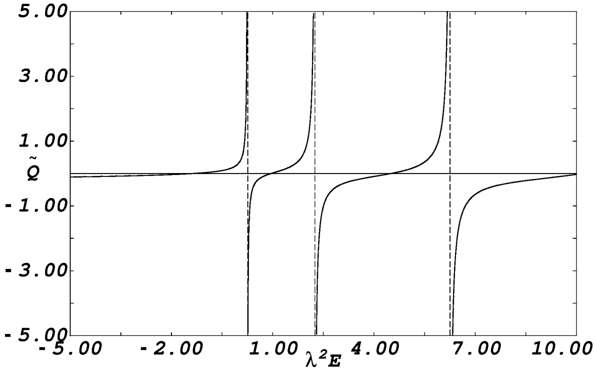


Fig. 4a $\tilde{Q}$ as a function of $\lambda^2 E$, $a/\lambda = 1$ (λ is fixed)

where $\nu = -\frac{1}{2} + a\sqrt{z}$. According to [23, 7.8(2) and 7.8(4)], we have

$$\lim_{\nu\to\infty} \mathcal{P}_\nu\left(\cos\frac{x}{\nu}\right) = J_0(x),$$
$$\lim_{\nu\to\infty} \mathcal{Q}_\nu\left(\cos\frac{x}{\nu}\right) = -\frac{\pi}{2}Y_0(x),$$

where J_0 and Y_0 are the Bessel functions. Therefore, (64) gives the limit

$$G^0(x, y; E) \to -\frac{1}{4}[iJ_0(\sqrt{E}\rho(x, y)) + Y_0(\sqrt{E}\rho(x, y))]$$

as $a \to +\infty$. Using [23, 7.2(7) and 7.2(15)] we get the result:

$$G^0(x, y; E) \to \frac{1}{2\pi}K_0(\sqrt{-E}\rho(x, y)).$$

4.1.1 Properties of the $\mathcal{Q}$-Function

Now we analyze the behavior of $Q(z; R)$. First of all, due to (60), the function $z \mapsto Q(z; R)$ has poles exactly at the points E, $E \geq 0$, obeying the condition

$$\pi a\sqrt{E} = \frac{\pi}{2} + l\pi, \quad l = 0, 1 \dots,$$

i.e. the poles are exactly at the points E_l^0. Denote by $I_0 = (-\infty, E_0^0)$, $I_1 = (E_0^0, E_1^0), \dots,$ $I_l = (E_{l-1}^0, E_l^0), \dots$; we have $\frac{\partial Q}{\partial z} > 0$ on each interval I_l (see Figs. 4a and 4b). Hence, for all $R \in \mathbb{R}$

$$\lim_{E\to E_l^0 - 0} Q(E; R) = +\infty, \qquad \lim_{E\to E_l^0 + 0} Q(E; R) = -\infty.$$

Using (60) we can obtain after cumbersome algebra the following asymptotic representation of $z \mapsto Q(z; R)$ in a vicinity of a given pole E_l^0:

$$Q(z; R) = \frac{1}{2\pi}\ln(2a) - \frac{1}{2\pi}\big[A_{-1}a^{-2}(z - E_l^0)^{-1} + A_0 + A_1a^2(z - E_l^0) + A_2a^4(z - E_l^0)^2 + O(a^6(z - E_l^0)^3)\big], \tag{65}$$

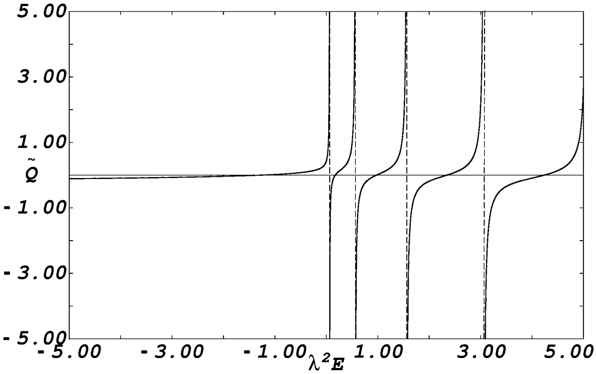


Fig. 4b $\tilde{Q}$ as a function of $\lambda^2 E$, $a/\lambda = 2$ (λ is fixed)

where

$$
\begin{aligned}
A_{-1} &= \frac{2l+1}{2},\\
A_0 &= \frac{1}{2(2l+1)} + \psi(l+1) + \gamma_E = 1 + \frac{1}{2} + \cdots + \frac{1}{l} + \frac{1}{2(2l+1)},\\
A_1 &= \frac{\psi'(l+1)}{2l+1} - \frac{1}{2(2l+1)^2} - \frac{\pi^2}{6}\\
&= -\frac{1}{2l+1}\left(1 + \frac{1}{4} + \cdots + \frac{1}{l^2} + \frac{1}{2(2l+1)} + \frac{l}{3}\pi^2\right),\\
A_2 &= \frac{\pi^2}{6} + \frac{1}{(2l+1)^2}\left(\frac{\psi''(l+1)}{2} + 1\right) - \frac{\psi'(l+1)}{(2l+1)^3}.
\end{aligned}
\tag{66}
$$

Let us find the asymptotics of Q as $a^2\mathrm{Re}z \to -\infty$. By (32) (or by (6.3.8) and (6.3.19) from [25])

$$
\psi\left(\frac{1}{2} + iz\right) + \psi\left(\frac{1}{2} - iz\right) = \ln(z^2) - \frac{1}{12z^2} - \frac{7}{480z^4} + O(z^{-6})
$$

as $|z| \to +\infty$, $z \notin i\mathbb{R}$. Therefore,

$$
Q(z; R) = -\frac{1}{2\pi}\left[\ln\sqrt{-z} - \ln 2 + \gamma + \frac{1}{24a^2 z} - \frac{7}{960a^4 z^2}\right] + O((a^2|z|)^{-3})) \tag{67}
$$

as $|z| \to \infty$, $z \notin \mathrm{spec}(H^0)$.

From (67) we recover again the limit (62); moreover,

$$
\lim_{E\to-\infty} Q(E; R) = -\infty.
$$

Using (41) we find an explicit expression for $Q(z; R)$ at the points $z = n^2/a^2$, $n \in \mathbb{N}$:

$$
Q\left(\frac{n^2}{a^2}; R\right) = -\frac{1}{\pi}\left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} - \frac{1}{2}\ln(8a)\right) \tag{68}
$$

(cf. with (36)).

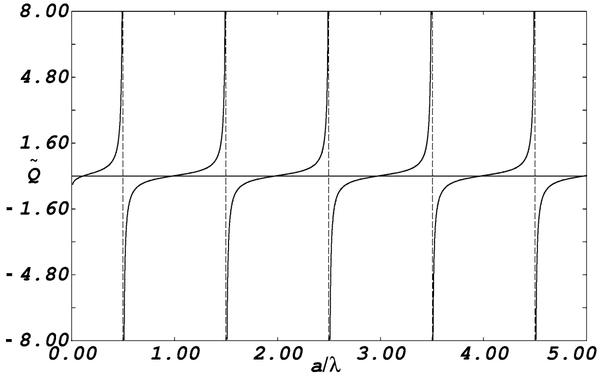


Fig. 5a $\tilde{Q}$ as a function of a/λ, $\lambda^2 E = 1$ (λ is fixed)

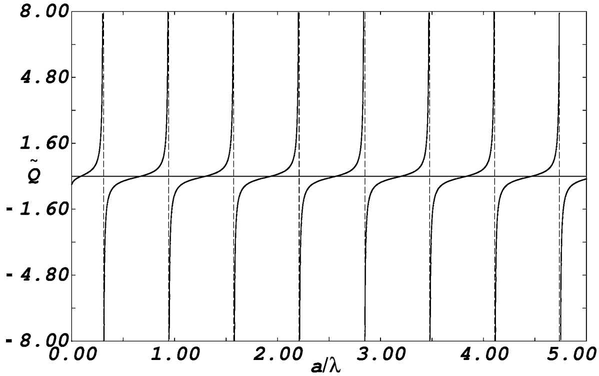


Fig. 5b $\tilde{Q}$ as a function of a/λ, $\lambda^2 E = 2.5$ (λ is fixed)

To investigate the behavior of the points levels as functions of λ and R, we establish some additional properties of Q. Using the standard series expansions of $\psi'(x)$, we can obtain the following expression

$$\frac{\partial Q}{\partial z}(z; R) = \frac{a^2}{4\pi} \sum_{l=0}^{\infty} \frac{2l+1}{((l+\frac{1}{2})^2 - a^2 z)^2}. \tag{69}$$

Equation (69) shows that $\partial Q/\partial z$ growth from 0 to $+\infty$ as z growth on the interval $I_0 = (-\infty, a^2/4)$. In particular, $\partial^2 Q/\partial z^2 > 0$ on this interval. As for the derivative with respect to a, we have

$$\frac{\partial Q}{\partial a}(z; R) = \frac{az}{2\pi} \left[\sum_{l=0}^{\infty} \frac{2l+1}{((l+\frac{1}{2})^2 - a^2 z)^2} + \frac{1}{a^2 z} \right]. \tag{70}$$

Equation (70) shows that $\frac{\partial Q}{\partial a}(E; R) > 0$, if $E > -[16a^2 \sum_{l\geq 0}(2l+1)^{-3}]^{-1}$. If $E < 0$ is fixed, then $\partial Q/\partial a$ changes its sign if a varies from 0 to ∞. Indeed, if $a \to 0$, then $1/a^2 E \to -\infty$, whereas the series in (70) tends to a finite limit $16 \sum_{l=0}^{\infty}(2l+1)^{-3}$. Therefore, if a is sufficiently small, then $\partial Q/\partial a > 0$. On the other hand, (67) shows that $a \mapsto Q(E; R)$ is a decreasing function for sufficiently large a, therefore, $\partial Q/\partial a < 0$. Similarly, if a is fixed, and $E < 0$, then for $|E|$ small enough we have $\partial Q/\partial a > 0$ (see above) and (67) shows again, that $\partial Q/\partial a < 0$ if $E \ll -1$. This behavior is illustrated by Figs. 5a, 5b, 6a and 6b.

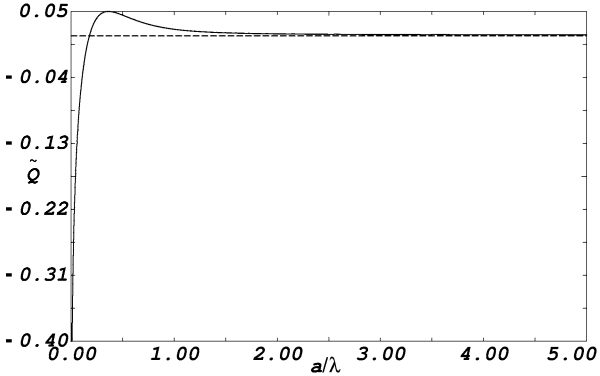


Fig. 6a $\tilde{Q}$ as a function of a/λ, $\lambda^2 E = -1$ (λ is fixed)

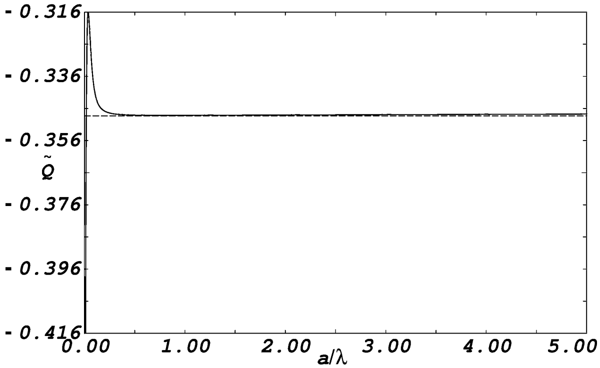


Fig. 6b $\tilde{Q}$ as a function of a/λ, $\lambda^2 E = -100$, (λ is fixed)

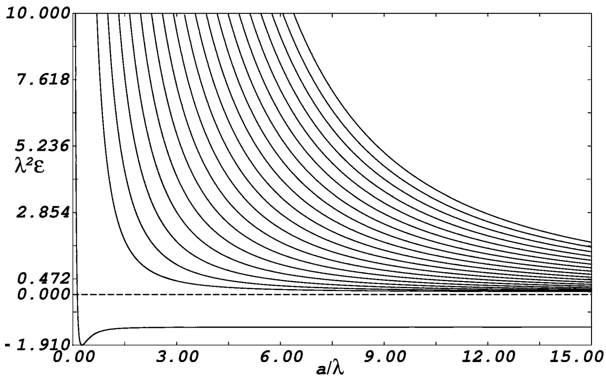


Fig. 7 Point levels $\lambda^2 \mathcal{E}_l^\lambda$ as a functions of a/λ for $l = 0, \ldots, 19$ (λ is fixed)

4.1.2 Behavior of the Point Levels

The results of the previous section show that for every $a > 0$ and $\alpha \in \mathbb{R}$ (i.e., for every $R > 0$ and $\lambda \in (0, +\infty)$) equation (16) has a unique solution (the point level) on each interval I_l, $l = 0, 1, \ldots$; denote this solution by $\mathcal{E}_l^\lambda(R)$. The plot of $\mathcal{E}^\lambda(R)$ is shown on Figs. 7, 8 and 9.

As in Sect. 3.1.2, it is convenient to rewrite equation (16)

$$Q(E; a) - \alpha = 0$$

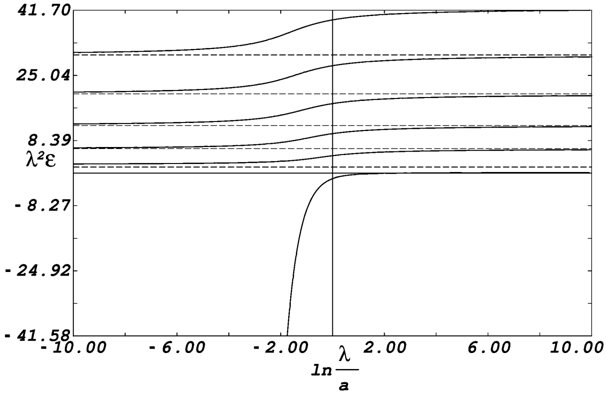


Fig. 8 Point levels $\lambda^2\mathcal{E}_l^\lambda$ as a functions of $\ln(\lambda/a)$ for $l = 0, \ldots, 5$ (a is fixed)

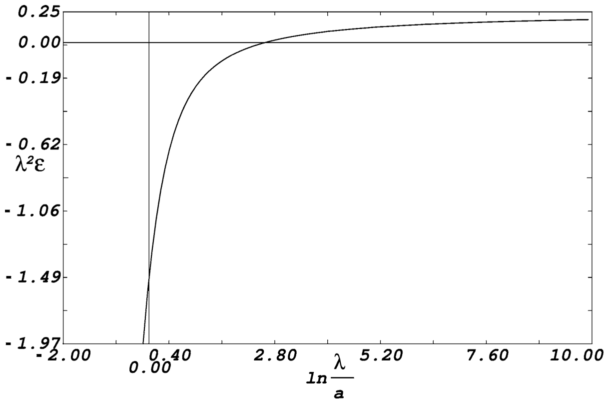


Fig. 9 Point level $\lambda^2\mathcal{E}_0^\lambda$ as a function of $\ln(\lambda/a)$ (a is fixed)

in the following form

$$\psi\left(\frac{1}{2} + a\sqrt{E}\right) - \frac{\pi}{2}\tan(\pi a\sqrt{E}) + \gamma = \ln\frac{2a}{\lambda}, \tag{71}$$

where the both sides are dimensionless. In the case $E < 0$, the following expression is more convenient

$$\mathrm{Re}\,\psi\left(\frac{1}{2} + a\sqrt{E}\right) + \gamma = \ln\frac{2a}{\lambda}. \tag{72}$$

The complete description of the spectrum of H^λ and behavior of its point levels is given by the following theorem.

Theorem 5

(1) *The spectrum of H^λ is purely discrete and consists of two non-intersecting series of levels*:
   (a) *Simple point levels $\mathcal{E}_l^\lambda(R)$, $l \geq 0$. The corresponding normalized eigenfunction has an explicit form* (17).
   (b) *Levels E_l^0, $l \geq 1$. Each level E_l^0 has the multiplicity $2l$ in the spectrum of H^λ, $\lambda \neq 0$. The corresponding eigensubspace coincides with the set $\{f \in L_l : f(q) = 0\}$, where L_l is the eigensubspace of H^0 which corresponds to E_l^0.*

(2) *At fixed* $R>0$ $\mathcal{E}_0^\lambda$ *increases from* $-\infty$ *to* $E_0^0 = 1/4a^2$ *and is concave when* $\lambda \in (0,+\infty)$, *each function* $\mathcal{E}_l^\lambda$, $l \ge 1$ *increases from* E_{l-1}^0 *to* E_l^0.
(3) *At fixed* $\lambda > 0$ $\mathcal{E}_l^\lambda$ *decreases with respect a if* $l \ge 1$

Proof (1) This statements is immediately follows from the results of the previous section.

(2) Equation (16) shows that at fixed $R>0$ $\mathcal{E}_0^\lambda$ increases from $-\infty$ to $E_0^0 = 1/4a^2$ as λ increasing from 0 to $+\infty$ and each function $\mathcal{E}_l^\lambda$, $l\ge 1$ increases from E_{l-1}^0 to E_l^0. Moreover,

$$\frac{\partial \mathcal{E}_l^\lambda}{\partial \lambda} = \frac{1}{2\pi\lambda}\left(\frac{\partial Q}{\partial z}\right)^{-1} > 0, \tag{73}$$

and

$$\frac{\partial^2 \mathcal{E}_0^\lambda}{\partial \lambda^2} = -\left[\frac{1}{2\pi\lambda^2} + \frac{\partial^2 Q}{\partial z^2}\left(\frac{\partial \mathcal{E}_0^\lambda}{\partial \lambda}\right)^2\right]\left(\frac{\partial Q}{\partial z}\right)^{-1} < 0. \tag{74}$$

In particular, $\lambda \mapsto \mathcal{E}_0^\lambda$ is a concave function.

(3) Since

$$\frac{\partial \mathcal{E}_l^\lambda}{\partial a} = -\frac{\partial Q}{\partial a}\left(\frac{\partial Q}{\partial z}\right)^{-1}, \tag{75}$$

we see that at fixed λ the level $\mathcal{E}_l^\lambda$ decreases with respect to a if $l \ge 1$. □

Now we investigate the asymptotic behavior of the point perturbed levels $\mathcal{E}_l^\lambda(R)$ of the operator H^λ.

Theorem 6

(1) *If* $\lambda \ll a$ *then* $a^2\mathcal{E}_0^\lambda(R) \to -\infty$ *and* $a^2\mathcal{E}_l^\lambda(R) \to (l+1/2)^2$ *if* $l \ge 1$. *Besides, the binding energy* $\mathcal{E}_1^\lambda(R) - \mathcal{E}_0^\lambda(R)$ *has the following estimate*:

$$\mathcal{E}_1^\lambda(R) - \mathcal{E}_0^\lambda(R) = -\mathcal{E}^\lambda(0) + \frac{1}{3a^2} + \frac{3}{2a^2}\left(\ln\frac{2a}{\lambda}\right)^{-1} + O\left(a^{-2}\left(\ln\frac{2a}{\lambda}\right)^{-2}\right),$$

which is greater than the binding energy $-\mathcal{E}^\lambda(0)$ *in the case of zero-curvature.*
(2) *If* $\lambda \gg a$ *then the binding energy* $\mathcal{E}_1^\lambda(a) - \mathcal{E}_0^\lambda(a)$ *have the estimate*:

$$\mathcal{E}_1^\lambda(R) - \mathcal{E}_0^\lambda(R) = \frac{1}{a^2}\left[2 + \left(\ln\frac{2a}{\lambda}\right)^{-1} + O\left(\left(\ln\frac{2a}{\lambda}\right)^{-2}\right)\right].$$

(3) *If* $a = 8\lambda$ *then* $\mathcal{E}_0^\lambda(R) = 0$ (*an appearance of zero-modes*). *Besides if*

$$\lambda_l = 8a\exp\left(-2\left(1+\frac{1}{3}+\cdots+\frac{1}{2l-1}\right)\right),$$

then $\mathcal{E}_l^{\lambda_l}(R) = l^2/a^2$ *and the behavior of* $\mathcal{E}_l^\lambda(R) \equiv \mathcal{E}_l(\alpha; R)$ *in a vicinity of the point* α_l *at fixed* R *is given by expansion*:

$$\mathcal{E}_l(\alpha; R) = \frac{l^2}{a^2} + B_l(\alpha-\alpha_l) + O((\alpha-\alpha_l)^2), \tag{76}$$

where a constant B_l depend of a and l. Moreover the behavior of $\mathcal{E}_l^\lambda(2/a^2)$ in the vicinity of a_l,

$$a_l = \frac{\lambda}{8}\exp\left(2+\frac{2}{3}+\cdots+\frac{2}{2l-1}\right),$$

is given by

$$\mathcal{E}_l^\lambda(2/a^2) = \frac{l^2}{a^2} + C_l(a-a_l) + O((a-a_l)^2), \tag{77}$$

where a constant C_l depend of a and l.

(4) *If a and λ are fixed, $l \to \infty$, then*

$$\mathcal{E}_l^\lambda(R) = \frac{1}{a^2}\left(l+\frac{1}{2}\right)^2 - \frac{1}{a^2}\left(l+\frac{1}{2}\right)\left[(\ln l)^{-1} + O((\ln l)^{-2})\right], \tag{78}$$

where only the term $O((\ln l)^{-2})$ depends on λ.

Proof (1) Starting from the ground state $\mathcal{E}_0^\lambda(R)$. In this case we have $a^2\mathcal{E}_0^\lambda(R) \to -\infty$. It is immediately clear if a is fixed and $\lambda \to 0$. Fix now λ and let $a \to \infty$. First, we remark that there is a_0 such that $\mathcal{E}_0^\lambda(R) < 0$ for each $a \geq a_0$. Suppose that there is a sequence $a_n \to \infty$ such that $\mathcal{E}_0^\lambda(R_n) \geq 0$, where $R_n = 2/a_n^2$. Since $\mathcal{E}_0^\lambda(R) < 1/4a^2$, we have $0 \leq a_n\sqrt{\mathcal{E}_0^\lambda(R_n)} < 1/2$; therefore (60) shows that $Q(\mathcal{E}_0^\lambda(R_n); R_n) \to \infty$. This contradicts to (16). Hence, the required a_0 exists. If there exists a sequence a_n, $a_n \to \infty$ such that the sequence $a_n\sqrt{\mathcal{E}_0^\lambda(R_n)}$ is bounded, then (60) shows again that $Q(\mathcal{E}_0^\lambda(R_n); R_n) \to \infty$.

Now we can use asymptotics (67) to get the following asymptotic representation for $\mathcal{E}_0$, which is similar to (45):

$$\mathcal{E}_0^\lambda(R) = \mathcal{E}^\lambda(0) - \frac{1}{12}a^{-2} - \frac{e^{2\gamma}}{360}\lambda^2 a^{-4} + O(\lambda^4 a^{-5}). \tag{79}$$

As for levels $\mathcal{E}_l^\lambda(R)$ with $l \geq 1$, we show first that $a^2\mathcal{E}_l^\lambda(R) \to (l+1/2)^2$. It is clear in the case when a is fixed and $\lambda \to 0$. Otherwise $\ln(2a) \to +\infty$, therefore (60) shows that $Q(\mathcal{E}_l^\lambda(R); R)$ remains constant only if $\tan(\pi a\sqrt{\mathcal{E}_l^\lambda(R)}) \to -\infty$. Hence, $a^2\mathcal{E}_l^\lambda(R) \to (l+1/2)^2$. Using (65) we get in the notations of (66) for $l \geq 1$

$$\begin{aligned}
\mathcal{E}_l^\lambda(R) &= E_{l-1}^0 + \frac{1}{a^2}\left[A_{-1}\left(\ln\frac{2a}{\lambda}\right)^{-1} + A_{-1}A_0\left(\ln\frac{2a}{\lambda}\right)^{-2}\right.\\
&\qquad \left. + (A_{-1}A_0^2 + A_{-1}^2A_1)\left(\ln\frac{2a}{\lambda}\right)^{-3} + O\left(\left(\ln\frac{2a}{\lambda}\right)^{-4}\right)\right]\\
&= E_{l-1}^0 + \frac{2l+1}{2a^2}\left[\left(\ln\frac{2a}{\lambda}\right)^{-1} + \left(1+\frac{1}{2}+\cdots+\frac{1}{l}+\frac{1}{2(2l+1)}\right)\left(\ln\frac{2a}{\lambda}\right)^{-2}\right.\\
&\qquad \left. + O\left(\left(\ln\frac{2a}{\lambda}\right)^{-3}\right)\right].
\end{aligned} \tag{80}$$

In particular, (79) and (80) show that in the considered case the binding energy $\mathcal{E}_1^\lambda(R) - \mathcal{E}_0^\lambda(R)$ has the following estimate:

$$\mathcal{E}_1^\lambda(R) - \mathcal{E}_0^\lambda(R) = -\mathcal{E}^\lambda(0) + \frac{1}{3a^2} + \frac{3}{2a^2}\left(\ln\frac{2a}{\lambda}\right)^{-1} + O\left(a^{-2}\left(\ln\frac{2a}{\lambda}\right)^{-2}\right).$$

(2) As above, we can show that in this case $\mathcal{E}_0^\lambda(R) \to E_0^0$. Therefore, we get from (65) (keeping the notations from (66))

$$\begin{aligned}\mathcal{E}_l^\lambda(R) &= E_l^0 + \frac{1}{a^2}\left[A_{-1}\left(\ln\frac{2a}{\lambda}\right)^{-1} + A_{-1}A_0\left(\ln\frac{2a}{\lambda}\right)^{-2}\right.\\ &\quad \left. + (A_{-1}A_0^2 + A_{-1}^2A_1)\left(\ln\frac{2a}{\lambda}\right)^{-3} + O\left(\left(\ln\frac{2a}{\lambda}\right)^{-4}\right)\right]\\ &= E_l^0 + \frac{2l+1}{2a^2}\left[\left(\ln\frac{2a}{\lambda}\right)^{-1} + \left(1 + \frac{1}{2} + \cdots + \frac{1}{l} + \frac{1}{2(2l+1)}\right)\left(\ln\frac{2a}{\lambda}\right)^{-2}\right.\\ &\quad \left. + O\left(\left(\ln\frac{2a}{\lambda}\right)^{-3}\right)\right] \qquad (81)\end{aligned}$$

for all $l \geq 0$.

So we have the estimate:

$$\begin{aligned}\mathcal{E}_1^\lambda(R) - \mathcal{E}_0^\lambda(R) &= E_1^0 - E_0^0 + \frac{1}{a^2}\left[\left(\ln\frac{2a}{\lambda}\right)^{-1} + O\left(\left(\ln\frac{2a}{\lambda}\right)^{-2}\right)\right]\\ &= \frac{1}{a^2}\left[2 + \left(\ln\frac{2a}{\lambda}\right)^{-1} + O\left(\left(\ln\frac{2a}{\lambda}\right)^{-2}\right)\right].\end{aligned}$$

(3) Let $\lambda = 8a$. Then using the (68), we get: $\mathcal{E}_0^\lambda(R) = 0$. Moreover, if we take

$$\lambda_l = 8a\exp\left(-2\left(1 + \frac{1}{3} + \cdots + \frac{1}{2l-1}\right)\right),$$

or, equivalently,

$$\alpha_l = \frac{\ln 8a}{2\pi} - \frac{1}{\pi}\left(1 + \frac{1}{3} + \cdots + \frac{1}{2l-1}\right),$$

then $\mathcal{E}_l^{\lambda_l}(R) = l^2/a^2$. Equations (76) and (77) follows from the Taylor expansion. In that formulas

$$B_l^{-1} = \begin{cases} \frac{la^2}{\pi}\sum_{k=0}^{l-1}(2k+1)^{-2}, & \text{if } l > 0;\\ \frac{7a^2}{2\pi}\zeta(3), & \text{if } l = 0,\end{cases}$$

where $\zeta(s)$ is the Riemann zeta-function ($\zeta(3) \approx .9203248$),

$$C_l = \begin{cases} -\frac{2l^2}{a_l^3} - \frac{1}{2la_l^3}\left(\sum_{k=0}^{l-1}(2k+1)^{-2}\right)^{-1}, & \text{if } l > 0;\\ -\frac{2l^2}{a_l^3} - \frac{1}{7a_l^3\zeta(3)}, & \text{if } l = 0.\end{cases}$$

(4) The proof of this item is clearly. □

4.2 Three-Dimensional Sphere: $d = 3$, $R > 0$

In this case $R = 6/a^2$, $a > 0$. Using the four-dimensional spherical coordinates [26], we have:

$$dl^2 = a^2 \left[d\chi^2 + \sin^2 \chi (\sin^2 \theta d\varphi^2 + d\theta^2) \right], \tag{82}$$

hence

$$\Delta_{LB} = \frac{1}{a^2} \left[\frac{\partial^2}{\partial \chi^2} + 2 \cot \chi \frac{\partial}{\partial \chi} + \frac{1}{\sin^2 \chi} L_2 \right], \tag{83}$$

where L_2 is the Laplace–Beltrami operator on the two-dimensional unit sphere (57). The Schrödinger operator has the form

$$H^0 = -\Delta_{LB} + \frac{1}{a^2}. \tag{84}$$

The spectrum of H^0 is purely discrete and consists of the levels (see, e.g., [24])

$$E_l^0 \equiv E_l^0(R) = \frac{1}{a^2}(l+1)^2; \quad l = 0, 1, \dots. \tag{85}$$

If $l \geq 1$, then level E_l^0 is degenerate with the multiplicity $(l+1)^2$. We will use the following expression for the Green function G^0 [15]:

$$\begin{aligned} G^0&(x, y; z) \\ &= \frac{1}{4\pi a \sin \frac{\rho(x,y)}{a}} \left[\cos(\rho(x, y)\sqrt{z}) - \sin(\rho(x, y)\sqrt{z}) \cot(\pi a \sqrt{z}) \right] \\ &= \frac{1}{4\pi a \sin \frac{\rho(x,y)}{a}} \left[\cosh(\rho(x, y)\sqrt{-z}) - \sinh(\rho(x, y)\sqrt{-z}) \coth(\pi a \sqrt{-z}) \right]. \end{aligned} \tag{86}$$

As $\rho(x, y) \to 0$, we have

$$G^0(x, y; z) = \frac{1}{4\pi \rho(x, y)} - \frac{1}{4\pi} \sqrt{z} \cot(\pi a \sqrt{z}) + O(\rho(x, y)), \tag{87}$$

therefore,

$$Q(z; R) = -\frac{1}{4\pi} \sqrt{z} \cot(\pi a \sqrt{z}) = -\frac{1}{4\pi} \sqrt{-z} \coth(\pi a \sqrt{-z}). \tag{88}$$

It is clear that for $z \in \mathbb{C} \setminus R_+$

$$Q(z; R) \to -\frac{1}{4\pi} \sqrt{z} = Q(z; 0), \tag{89}$$

as $R \to 0$. Moreover, if $z \in \mathbb{R}$ and $z < 0$, then

$$G^0(x, y; z) \to \frac{\exp(-\rho(x, y)\sqrt{-z})}{4\pi \rho(x, y)}$$

as $a \to +\infty$; in other words, $G^0(x, y; z)$ tends to the Green function of the free Schrödinger operator in Euclidean space.

4.2.1 Properties of the Q-Function

From (88) we see that $z \mapsto Q(z; R)$ has poles exactly at points E, $E \geq 0$, obeying the condition

$$\pi a \sqrt{E} = \pi l, \quad l = 1, 2, \ldots,$$

i.e. the poles are exactly at points E_l^0. Denote $I_0 = (-\infty, E_0^0)$, $I_1 = (E_0^0, E_1^0), \ldots, I_l = (E_{l-1}^0, E_l^0), \ldots$.

The next theorem describe behavior of the Q-function.

Theorem 7

(1) *$Q(z; R)$ is increasing function of z on each interval I_l. Moreover, for $E \in \mathbb{R}$*

$$\lim_{E \to E_l^0 - 0} Q(E; R) = +\infty, \qquad \lim_{E \to E_l^0 + 0} Q(E; R) = -\infty.$$

(2) *The behavior of $Q(z; R)$ near a pole E_l^0 is given by expansion*

$$Q(z; R) = -\frac{(l+1)^2}{2\pi^2 a^3}(z - E_l^0)^{-1} - \frac{3}{8\pi^2 a} + \frac{a}{8}\left(\frac{1}{3} + \frac{1}{4\pi^2 (l+1)^2}\right)(z - E_l^0) + O((z - E_0^l)^2). \tag{90}$$

The behavior of $Q(z; R)$ as $z < 0$ and $a^2|z| \to \infty$ is given by asymptotics

$$Q(z; R) = -\frac{1}{4\pi}\sqrt{-z} - \frac{1}{2\pi}\sqrt{-z}\exp(-2\pi a\sqrt{-z}) + O(\sqrt{-z}\exp(-4\pi a\sqrt{-z})), \tag{91}$$

(3) *$E \mapsto Q(E; R)$ is a convex function on the interval $(-\infty, a^{-2})$.*
(4) *$Q(E; a)$ is an increasing function of a for all $E \in \mathbb{R} \setminus \mathrm{spec}(H^0)$ and $a > 0$.*

Proof (1) This item follows from the explicit expression for derivative of Q-function

$$\frac{\partial}{\partial z} Q(z; R) = \frac{1}{8\pi\sqrt{z}} \frac{2\pi a\sqrt{z} - \sin(2\pi a\sqrt{z})}{2\sin^2(\pi a\sqrt{z})} = \frac{1}{8\pi\sqrt{-z}} \frac{\sinh(2\pi a\sqrt{-z}) - 2\pi a\sqrt{-z}}{2\sinh^2(\pi a\sqrt{-z})}. \tag{92}$$

(2) First expansion follows from the Taylor expansion. Second expansion is the result of using asymptotic

$$\coth x = 1 + 2e^{-2x} + O(e^{-4x}),$$

as $x \to +\infty$.

(3) Let us show that

$$\frac{\partial^2}{\partial z^2} Q(E; R) > 0 \tag{93}$$

if $-\infty < E < E_0^0 = 1/a^2$. On the interval $(-\infty, 0)$ we have

$$Q(E;R) = -\frac{1}{4\pi a^2} f(\pi a\sqrt{-E}),$$

where $f(x) = x\coth x$, $0 < x < \infty$. Since $f'(x) = \frac{2\sinh\frac{x}{2} - x}{\sinh^2 x} > 0$, the function f increases. Since $\sqrt{-E}$ is a concave function of E, to prove $E \mapsto Q(E;R)$ is convex it is sufficient to prove f is convex. This follows from

$$\begin{aligned} f''(x) &= \frac{2}{\sinh^3 x}(x\cosh x - \sinh x) \\ &= \frac{2}{\sinh^3 x}\left[\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n)!} - \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\right] > 0, \end{aligned}$$

if $x > 0$.

Now we consider the interval $0 \le E < 1/a^2$. According to (92), to prove that $E \mapsto Q(E;R)f$ is convex on this interval it is sufficiently to prove that on the interval $[0,\pi)$ we have $g'(x) > 0$, where

$$g(x) = \frac{1}{\sin^2 x} - \frac{\cot x}{x},$$

and that $g'(x) = Ax + O(x^2)$ as $x \to 0$, where $A > 0$. Since

$$g'(x) = \frac{\cos x\sin^2 x + x\sin x - 2x^2\cos x}{x^2\sin^3 x},$$

we calculate $g'(x) = \frac{3}{4}x + O(x^3)$ and it remains to prove that $h(x) > 0$ on the interval $(0,\pi)$, where $h(x) = \cos x\sin^2 x + x\sin x - 2x^2\cos x$. If $\pi/2 \le x < \pi$, the required inequality follows immediately from $h(x) = \cos x(\sin^2 x - 2x^2) + x\sin x$. Since $h(0) = 0$ it is sufficient to prove that $h'(x) > 0$ if $0 < x < \pi/2$. We have $h'(x) = 3\sin x\cos^2 x - 3x\cos x + 2x^2\sin x$. We see that $h'(0) = 0$ and we prove that $h''(x) > 0$ on $(0,\pi/2)$. Indeed, $h''(x) = 9\sin x(x - \frac{1}{2}\sin 2x) + 2x^2\cos x$, therefore, $h''(x) > 0$. Thus, function $E \mapsto Q(E;R)$ is convex on the interval $(-\infty, a^{-2})$.

(4) Let us calculate $\frac{\partial Q}{\partial a}$:

$$\frac{\partial}{\partial a}Q(E;a) = \begin{cases} \frac{E}{4\sin^2\pi a\sqrt{E}}, & \text{if } E > 0; \\ \frac{-E}{4\sinh^2\pi a\sqrt{-E}}, & \text{if } E < 0; \\ \frac{1}{4\pi a^2}, & \text{if} E = 0. \end{cases} \tag{94}$$

Hence, $\frac{\partial}{\partial a}Q(E;a) > 0$ for all $E \in \mathbb{R}\setminus \operatorname{spec}(H^0)$ and $a > 0$. □

For the plotting of Q-function $Q(E;R)$ we use dimensionless quantity λQ as below. This plot is shown on Figs. 10 and 11.

4.2.2 Behavior of the Point Levels

The results of the previous section show that for every $a > 0$ and $\alpha \in \mathbb{R}$ (i.e., for every $R > 0$ and $\lambda \in (0,+\infty)$) equation (16) has a unique solution (the point level) on each interval I_l,

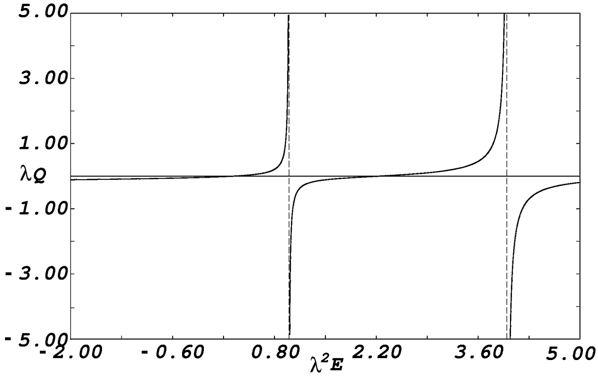


Fig. 10a λQ as a function of $\lambda^2 E$, $a/\lambda = 1$ (λ is fixed)

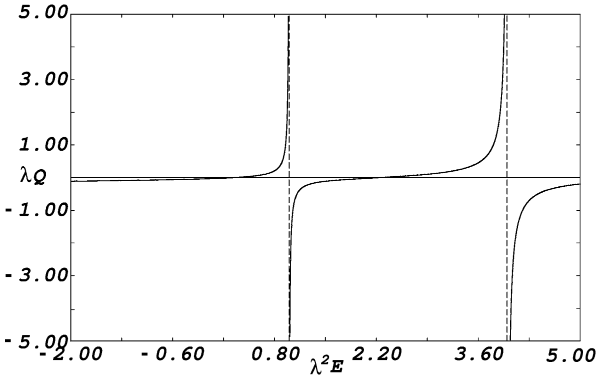


Fig. 10b λQ as a function of $\lambda^2 E$, $a/\lambda = 2$ (λ is fixed)

$l = 0, 1, \ldots$; denote this solution by $\mathcal{E}_l(R)$. The plot of $\mathcal{E}^\lambda(R)$ is shown on Figs. 12,13 and 14.

As in the previous sections, it is convenient to rewrite (16)

$$Q(E; a) - \alpha = 0$$

in the following form

$$\lambda\sqrt{z}\cot\pi a\sqrt{z} = 1. \tag{95}$$

In the case $E < 0$, the following expression is more convenient

$$\lambda\sqrt{-z}\coth\pi a\sqrt{-z} = 1. \tag{96}$$

Curiously that (95) and (96) coincide with equations for the spectrum of the one-dimensional Schrödinger operator with a periodic δ'-potential at quasimomentum $p = 0$, if the period of the potential is $2\pi a$ and the strength is -2λ (see [12, (III.3.37)]).

Description of the spectrum of H^λ and properties of it's point levels is given by the next theorem.

Theorem 8

(1) *The spectrum of H^λ is purely discrete and consists of two non-intersecting series of levels*:

Fig. 11a λQ as a function of a/λ, $\lambda^2 E = 2$ (λ is fixed)

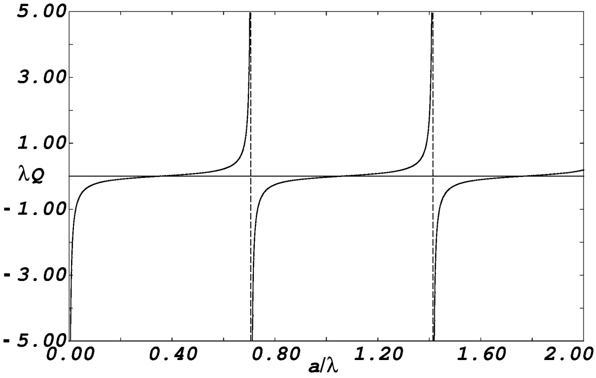


Fig. 11b λQ as a function of a/λ, $\lambda^2 E = -1$ (λ is fixed)

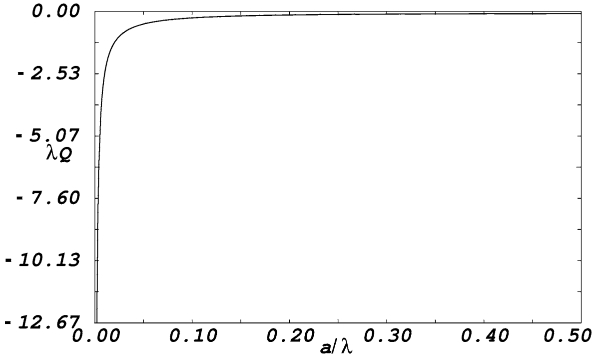
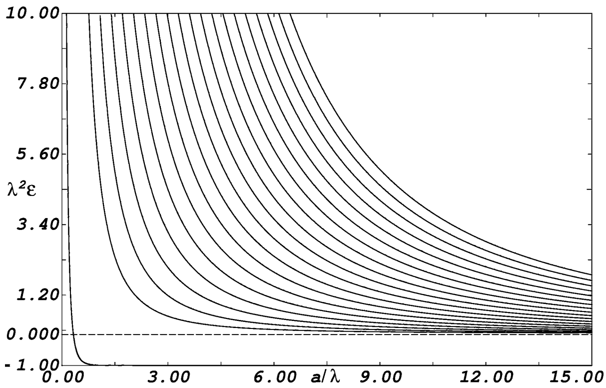


Fig. 12 Point levels $\lambda^2 \mathcal{E}_l^\lambda$ as a functions of a/λ for $l = 0, \dots, 19$ (λ is fixed)



(a) *Simple point levels* $\mathcal{E}_l^\lambda(R)$, $l \geq 0$. *The corresponding normalized eigenfunction has an explicit form* (17).

(b) *Levels* E_l^0, $l \geq 1$. *Each level* E_l^0 *has in the spectrum of* H^λ, $\lambda \neq 0$, *the multiplicity* $l^2 + 2l$. *The corresponding eigensubspace coincides with the set* $\{f \in L_l : f(q) = 0\}$, *where* L_l *is the eigensubspace of* H^0 *which corresponds to* E_l^0.

(2) *At fixed* $R > 0$, $\mathcal{E}_0^\lambda$ *increases from* $-\infty$ *to* $E_0^0 = 1/a^2$ *as* α *increases from* $-\infty$ *to* $+\infty$ *and each function* $\mathcal{E}_l$, $l \geq 1$ *increases in this case from* E_{l-1}^0 *to* E_l^0. *Besides* $\alpha \mapsto \mathcal{E}_0^\lambda$ *is a concave function of* α.

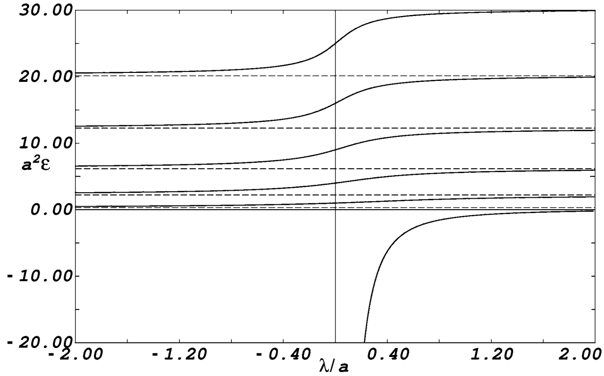


Fig. 13 Point levels $\lambda^2\mathcal{E}_l^\lambda$ as a functions of λ/a for $l = 0, \dots, 4$ (a is fixed)

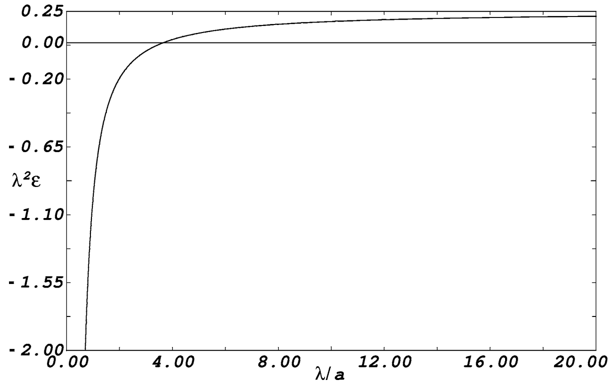


Fig. 14 Point level $\lambda^2\mathcal{E}_0^\lambda$ as a function of λ/a (a is fixed)

(3) *At fixed α, Each level $\mathcal{E}_l^\lambda(R)$ is an increasing function of R.*

Proof (1) Statements of this item is simply followed from the properties of the $\mathcal{Q}$-function, which is described at the previous section.

(2) Equation (16) immediately shows that at fixed R, $\mathcal{E}_0^\lambda$ increases from $-\infty$ to $E_0^0 = 1/a^2$ as α increases from $-\infty$ to $+\infty$ and each function $\mathcal{E}_l$, $l \geq 1$ increases in this case from E_{l-1}^0 to E_l^0. Moreover,

$$\frac{\partial \mathcal{E}_l^\lambda}{\partial \alpha} = \left(\frac{\partial Q}{\partial z}\right)^{-1} > 0, \tag{97}$$

and

$$\frac{\partial^2 \mathcal{E}_0^\lambda}{\partial \alpha^2} = -\frac{\partial^2 Q}{\partial z^2}\left(\frac{\partial Q}{\partial z}\right)^{-3} < 0. \tag{98}$$

In particular, $\alpha \mapsto \mathcal{E}_0^\lambda$ is a concave function of α.

(3) This item is the result of using (75) and (94). □

The behavior of the point levels at three limiting cases is described by the following theorem.

Theorem 9

(1) *If* $\lambda \sim a$ *then* $\mathcal{E}_0^\lambda(R) = 0$ (H^λ *has zero-modes*), *if and only if* $\lambda = \pi a$. *In a vicinity of* $\alpha_0 \geq -\frac{1}{4\pi^2 a}$ *we have*

$$\mathcal{E}_0(\alpha; R) = \frac{12}{a}(\alpha - \alpha_0) + O((\alpha - \alpha_0)^2). \tag{99}$$

(2) *If* $|\lambda| \gg a$:

(2a) *if* a *is fixed*, $\lambda = \infty$, *then*

$$\mathcal{E}_l^\infty(R) = \frac{1}{a^2}\left(l + \frac{1}{2}\right)^2, \quad l \geq 0; \tag{100}$$

(2b) *if* a *is fixed*, $\lambda \to \infty$, *then*

$$\mathcal{E}_l^\lambda(R) = \frac{1}{a^2}\left(l + \frac{1}{2}\right)^2 - \frac{2}{\pi\lambda a} + O(\lambda^{-2}); \tag{101}$$

(2c) *if* $a \to 0$, $\lambda \neq 0$ *is fixed, then*

$$\mathcal{E}_l^\lambda(R) = \frac{1}{a^2}\left(l + \frac{1}{2}\right)^2 - \frac{2}{\pi\lambda a} + O(1). \tag{102}$$

(3) *If* $|\lambda| \ll a$:

(3a) *if* a *is fixed*, $\lambda \to +0$, *then* $\mathcal{E}_0^\lambda(R) \to -\infty$ *and*

$$\begin{aligned}\mathcal{E}_0^\lambda(R) &= \mathcal{E}_0^\lambda(0) - 4\mathcal{E}_0^\lambda(0)\exp\big(-2\pi a\sqrt{-\mathcal{E}_0^\lambda(0)}\big) + O\big(\exp(-4\pi a\sqrt{-\mathcal{E}_0^\lambda(0)}\big)\\ &= -\lambda^{-2} + 4\lambda^{-2}\exp(-2\pi a\lambda^{-1}) + O(\exp(-4\pi a\lambda^{-1})).\end{aligned} \tag{103}$$

For $l \geq 1$

$$\begin{aligned}\mathcal{E}_l^\lambda(R) &= E_{l-1}^0(R) - \frac{l^2}{a^2}\left(\frac{1}{2\pi^2\alpha} - \frac{3}{16\pi^4\alpha^2} + O(|a\alpha|^{-3})\right)\\ &= \frac{l^2}{a^2}\left(1 + \frac{2}{\pi}\frac{\lambda}{a} + \frac{3}{\pi^2}\frac{\lambda^2}{a^2} + O\left(\frac{\lambda^3}{a^3}\right)\right).\end{aligned} \tag{104}$$

(3b) *if* a *is fixed*, $\lambda \to -0$, *then*

$$\begin{aligned}\mathcal{E}_l^\lambda(R) &= E_l^0(R) - \frac{(l+1)^2}{a^2}\left(\frac{1}{2\pi^2\alpha} - \frac{3}{16\pi^4\alpha^2} + O(|a\alpha|^{-3})\right)\\ &= \frac{(l+1)^2}{a^2}\left(1 + \frac{2}{\pi}\frac{\lambda}{a} + \frac{3}{\pi^2}\frac{\lambda^2}{a^2} + O\left(\frac{\lambda^3}{a^3}\right)\right)\end{aligned} \tag{105}$$

for all $l \geq 0$.

(3c) *if* $a \to \infty$, $\lambda \neq 0$ *is fixed, then* $a\sqrt{-\mathcal{E}_0^\lambda(R)} \to \infty$ *and*

$$\mathcal{E}_l^\lambda(R) = \frac{1}{a^2}\left(l + \frac{1}{2}\right)^2 - \frac{2}{\pi\lambda a} + O(1). \tag{106}$$

(4) *If a and λ are fixed, $l \to \infty$, each level $\mathcal{E}^{\lambda}$ behaves as $\mathcal{E}^{\infty}$ with a constant correction*:

$$\mathcal{E}_l^{\lambda}(R) = \frac{1}{a^2}\left(l + \frac{1}{2}\right)^2 - \frac{2}{\pi\lambda a} + O(l^{-2}), \quad \text{as } l \to \infty. \tag{107}$$

The proof of this theorem is simply followed from the results which is given earlier.

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